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PROGRESS REPORT.

The following gentlemen have been elected members of the Society.

1. Mr. S. V. Venkataraya Sastri B.A., L.T., Mathematics Teacher, the College, Kottayam (*Concession rate*);
2. Mr. Kerala Varma, the 23rd Prince of Cochin, Cochin House, Cathedral Post, Madras; and
3. Capt. A. G. McKendrich, I.M.S., Assistant Director of Pasteur Institute, Coonoor.

2. The following books have been received as presents to the Library:—

1. Madras University Calendar, Vols. I, II, and Examination Papers.
2. Co-ordinate Geometry of Three Dimensions: by Robert J. T. Bell (*Macmillan & Co., Ltd.*) 10*s. net.*
3. Arithmetique Generale (*Nombres naturels, qualifiers, complexes, ternions and quaternions*): by Prof. E. Dumont (*A. Herman and Fils*) 10 francs.
4. Theorie des Series a termes & constants: by Prof. E. Fabry (*A. Herman and Fils*) 6·50 francs.
5. Introduction a la theorie des nombres Algebriques (*from the German of Dr. J. Sommer of Dantzic, with notes and additions*): by Prof. A. Levy. (*A. Herman and Fils*) 15 francs.
6. Systemes Cinematiques (*Scientia Series No.* 31): by Dr. L. Crelier (*Ganthier Villars*) 2 francs.

3. The following books have been purchased for the Library—

1. Introduction a la Theorie des Functions d'une variable Vol. I —by J. Tannery (1904), 14 francs.
2. „ „ Vol. II—by J. Tannery (1910), 15 francs.
3. Principia Mathematica—by Whitehead and Russell (1910) 25*s.*
4. History of the Theories of Ether and Electricity—by E. T. Whittaker (1910) 12*s.* 6*d.*
5. Researches Arithmetiques—by C. F. Gauss (1910) 13*s.*
6. Theorie des Nombres—by E. Lucas (1891) 12*s.*
7. Vector Analysis—by E. B. Wilson (1909) 16*s.* 6*d.*
8. Principles of Graphic Statics—by Major G. S. Clarke (1888) 7*s.* 6*d.*

POONA,
1st June 1911.

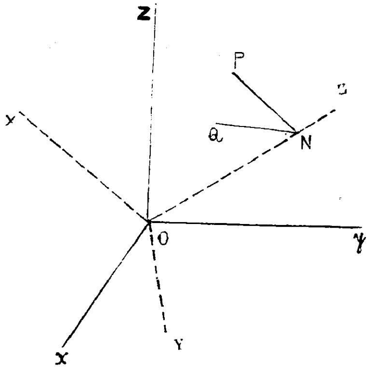
D. D. KAPADIA,
Hon. Joint Secretary.

An Introduction to the theory of moving axes with application to curves in space and curves on surfaces.

By A. C. L. Wilkinson, M.A., F.R.A.S.

It has been suggested to me that some articles dealing with mathematical subjects somewhat in advance of the standard English text books will be welcomed by those readers of the Indian Mathematical Journal who have not access to the great modern French and German treatises on Pure Mathematics. I have chosen the subject of moving axes with their application to the problems of Solid Geometry as no account of this important analytical method has yet found a place in any of the English works on the subject. Routh's pioneer paper in the Cambridge Mathematical Journal of 1836 is probably not available to any readers of this Journal and I myself only possess some short notes taken from it many years ago, A very brief reference is given in Vol. II of Routh's Rigid Dynamics §, 21. Darboux's monumental work " Theorie des Surfaces" is based entirely on moving axes. The method and application I give, differ from those of Darboux in as much as I do not consider the general system based on two independent parameters, and further I develop the theory entirely independently of all kinematical considerations. The first part of this paper will deal with properties of curves in space, the second with curves on surfaces of revolution and triply orthogonal surfaces including the confocal conicoids, and the third with curves on surfaces in general. The examples have been taken from various examination papers. Goursat Traité d'Analyse, Tome I, Resal Théorie des Surfaces, Proff. V and K Kommerell Allgemeine Theorie dez Raumcurven and Flächen and Darboux Theorie des Surfaces.

§ 1. Take three rectangular axes Ox, Oy, Oz, forming a right handed system, so that the positive directions of rotation about Ox, Oy, Oz are from y to z, z to x, x to y. By a positive rotation, I mean the rotation of a right handed screw advancing in the positive direction of the axis; or when looking along the positive direction of the axis at the face of a watch, turned towards the observer, the positive direction of rotation is the direction in which the hands of the watch turn.



Consider a rotation, in the positive sense, through an angle θ about an axis OZ whose direction cosines are ($\cos\alpha$, $\cos\beta$, $\cos\gamma$). We wish to find the coordinates of Q, the point to which a given point P (x, y, z) is displaced by the rotation. Draw PN perpendicular to OZ, then the coordinates of N are ($k\cos\alpha$, $k\cos\beta$, $k\cos\gamma$) where $k=$ ON $=x\cos\alpha+y\cos\beta+z\cos\gamma$.

Take a new set of rectangular axes as follows :

OX parallel to NP and in the direction N to P :

OZ the axis of rotation :

OY perpendicular to the plane OXZ,

so that OX, OY, OZ also form a right handed system.

Denote $OP^2=x^2+y^2+z^3$ by r^2.

The direction cosines of OX, OY, OZ are

$$\frac{x-k\cos\alpha}{\sqrt{r^2-k^2}},\quad \frac{y-k\cos\beta}{\sqrt{r^3-k^2}},\quad \frac{z-k\cos\gamma}{\sqrt{r^2-k^2}},$$

$$\pm\frac{z\cos\beta-y\cos\gamma}{\sqrt{r^2-k^2}},\quad \pm\frac{x\cos\gamma-z\cos\alpha}{\sqrt{r^2-k^2}},\quad \pm\frac{y\cos\alpha-x\cos\beta}{\sqrt{r^3-k^2}},$$

$$\cos\alpha,\qquad \cos\beta,\qquad \cos\gamma.$$

The sign of the ambiguity may be determined from the consideration that the determinant formed by the nine direction cosines, as written, must be $+1$, Thus the $+$ sign of the ambiguity must be taken. Otherwise, we may consider the special position of the axes got by putting:

$\cos\alpha=0$, $\cos\beta=0$, $\cos\gamma=1$, $x=0$, $u=-a$, $z=0$. $r=a$, $k=0$,

when the axes OX, OY, OZ become (0,—1, 0), (1, 0, 0), (0, 0, 1), again giving the + sign of the ambiguity.

Now the coordinates of P referred to OX, OY, OZ are

$$(\sqrt{r^2-k^2}, 0, k)$$

and the coordinates of Q are

$$(\sqrt{r^2-k^2}\cos\theta, \sqrt{r^2-k_z}\sin\theta, k)$$

Hence the coordinates of Q when referred to Ox, Oy, Oz will be

$$\frac{x-k\cos\alpha}{\sqrt{r^2-k^2}}\sqrt{r^2-k^2}\cos\theta+\frac{z\cos\beta-y\cos\gamma}{\sqrt{r^2-k^2}}\sqrt{r^2-k^2}\sin\theta+k\cos\alpha,$$

and similar expressions for the y and z coordinates.
These reduce to

$$(x-k\cos\alpha)\cos\theta+(z\cos\beta-y\cos\gamma)\sin\theta+k\cos\alpha,$$
$$(y-k\cos\beta)\cos\theta+(x\cos\gamma-z\cos\alpha)\sin\theta+k\cos\beta,$$
$$(z-k\cos\gamma)\cos\theta+(y\cos\alpha-x\cos\beta)\sin\theta+k\cos\gamma,$$

where $k=x\cos\alpha+y\cos\beta+z\cos\gamma$.

If now we consider an infinitesimal rotation, so that $\sin\theta$, $\cos\theta$ can be replaced by θ, 1 respectively, the above coordinates of Q reduce to

$$x+\theta(mz-ny),\ y+\theta(nx-lz),\ z+\theta(ly-mx)$$

where (l, m, n) are the direction cosines of the axis of rotation and θ the angle, supposed infinitesimal, rotated through.
These are finally written

$$x+z\partial_2-y\partial_3,\ y+x\partial_3-z\partial_1,\ z+y\partial_1-x\partial_2,\ \dots\dots\dots(1)$$

where ∂_1, ∂_2, ∂_3 are written for $l\theta$, $m\theta$, $n\theta$ and may be called the components of the rotation.

§. 2. Now consider a set of rectangular axes moving in space as follows: (1) an infinitesimal rotation about a straight line through the origin represented by the components ∂_1, ∂_2, ∂_3; (2) an infinitesimal translation of the origin represented by ξ, η, ζ. Consider a point P (x, y, z) referred to the first set of axes and a consecutive point P′ whose coordinate are $(x+dx, y+dy, z+dz)$ referred to the second set of axes defined above. If the coordinates of P′ referred to the first set of axes are $(x+\Delta x, y+\Delta y, z+\Delta z)$ we require to express Δx, Δy, Δz in terms of dx, dy, dz and the quantities that determine the position of the second set of axes.

The result (1) of § 1 can be expressed as follows: a point whose coordinates are (x, y, z) referred to the second system of axes is when

referred to the first system $(x+z\partial_1-y\partial_2,\ y+x\partial_3-z\partial_1,\ z+y\partial_1-x\partial_2)$, when no translation of the axes occurs.

Neglecting infinitssimals of the second order and taking account of the translation ξ, η, ζ, the coordinates of P′, referred to the first system of axes, are

$$(x+dx+\xi-y\partial_3+z\partial_2,\ y+dy+\eta-z\partial_1+x\partial_3,\ z+dz+\zeta-x\partial_2+y\partial_1).$$

If Δs is the element of arc of the locus of P, we have

$$\Delta s^2=\Sigma\,(dx+\xi-y\partial_3+z\partial_2)^2. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \ (1)$$

The direction cosines of the tangent at P to its locus are

$$\frac{dx+\xi-y\partial_3+z\partial_2}{\Delta s},\ \frac{dy+\eta-z\partial_1+x\partial_3}{\Delta s},\ \frac{dz+\zeta-x\partial_2+y\partial_1}{\Delta s} \quad \ldots \ (2)$$

These equations are fundamental in the theory of moving axes.

§. 3 *If (l, m, n) are the direction cosines of a straight line referred to the first position of a set of moving axes, $(l+dl, m+dm, n+dn)$ the direction cosines of the consecutive position of the straight line referred to the consecutive position of the axes, to find the angle between these straight lines.*

The points (o, o, o) $(l+dl,\ m+dm,\ n+dn)$ referred to the second position of the axes are when referred to the first position of the axes

$$(\xi, \eta, \zeta) \text{ and } (\xi+l+dl-m\partial_3+n\partial_2, \ldots, \ldots)$$

Therefore the direction cosines of the consecutive position of (l,m,n) referred to the first set of axes are

$$(l+dl-m\partial_3+n\partial_2\,,\ m+dm-n\partial_1+l\partial_3\,,\ n+dn-l\partial_2+m\partial_1) \quad \ldots \quad (1)$$

If $\Delta\varepsilon$ is the angle between these straight lines

$$\Delta\varepsilon^2=(dl-m\partial_3+n\partial_2)^2+(dm-n\partial_1+l\partial_3)^2+(dn-l\partial_2+m\partial_1)^2 \quad \ldots \quad (2)$$

Thus the radius of curvature of the locus of P is given by

$$\rho=\frac{\Delta s}{\Delta\varepsilon} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where Δs is given by §. 2, (1) and (l, m, n) are the direction cosines of the tangent given by §. 2, (3).

The expression (2) can be obtained as follows:

(l, m, n), $(l+\Delta l,\ m+\Delta m,\ n+\Delta n)$ being the direction cosines of two lines indefinitely near to each other

$$l^2+m^2+n^2=1$$

$$(l+\Delta l)^2+(m+\Delta m)^2+(n+\Delta n)^2=1$$

whence : $$2\,(l\Delta l+m\Delta m+n\Delta n)+\Delta l^2+\Delta m^2+\Delta n^2=0$$

and $$\cos\theta=1-\frac{\theta^2}{2}=l\,(l+\Delta l)+m\,(m+\Delta m)+n(n+\Delta n)$$

$$\therefore\ \theta^2=\Delta l^2+\Delta m^2+\Delta n^2.$$

To determine the radius of torsion of the locus of P.

The direction cosines (λ, μ, ν) of the binormal are got by solving the equations

$$l\lambda + m\mu + n\nu = 0$$

$$\lambda\,(dl - m\,\partial_3 + n\,\partial_2) + \mu\,(dm - n\partial_1 + l\partial_3) + \nu\,(dn - l\partial_2 + m\partial_1) = 0$$

If ΔT is the angle of torsion, we have

$$\Delta T^2 = \Sigma\,(d\lambda - \mu\,\partial_3 + \nu\partial_2)^2 \quad \ldots \quad \ldots \quad \ldots (4)$$

and for σ the radius of torsion, we have

$$\frac{1}{\sigma} = \frac{\Delta T}{\Delta s} \quad \ldots \quad \ldots \quad \ldots \quad \ldots (5)$$

A difficulty will sometimes arise in regard to the signs of ρ, σ. In general the determination of the positive direction of the principal normal will give no difficulty and then ρ will be taken positively, the determination of the sign of σ is usually difficult and in the problems I consider I shall not enter into the relative signs of the torsions of the related curves.

Finally we must consider the relation of two consecutive planes.

If $ax + by + cz + p = 0$ is the equation of a variable plane referred to the first set of axes, the consecutive plane referred to the second set of axes will be :

$$(a + da)\,X + (b + db)\,Y + (c + dc)\,Z + p + dp = 0.$$

A point X, Y, Z referred to the second set of axes is, when referred to the first set, (x, y, z) where by §. 2,

$$x = X + \xi - Y\partial_3 + Z\partial_2, \text{ etc.}$$

Solving these equations for X, Y, Z and neglecting the infinitesimals of the second order, the consecutive plane referred to the first set of axes is :

$$a\,(x - \xi + y\partial_3 - z\partial_2) + b\,(y - \eta + z\partial_1 - x\partial_3) \quad \ldots (6)$$
$$+ c\,(z - \zeta + x\partial_2 - y\partial_1) + p + dp + xda + ydb + zdc = 0$$

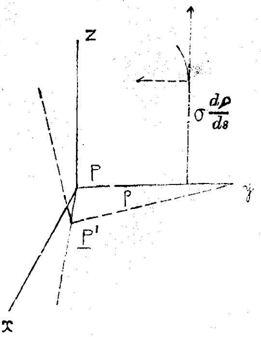
§. 4. To consider the set of moving axes formed by the tangent, normal and binormal to a curve in space whose angles of contingence and torsion are $d\epsilon$, dT : the axes forming a right handed system, which thus determines the positive direction of the binormal.

The tangent, normal and binormal are :

$$(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1).$$

The consecutive positions of these straight lines are by § 3, (1)

$$(1, \partial_3, -\partial_2),\ (-\partial_3, 1, \partial_1),\ (\partial_2, -\partial_1, 1).$$



The binormal (0, 0, 1) is perpendicular to the consecutive tangent $(1, \partial_3, -\partial_2)$, hence $\partial_1 = 0$.

The angles between the consecutive tangents and consecutive binormals are respectively $d\varepsilon$, dT: hence

$$\partial_3 = d\varepsilon,\ \partial_1 = dT.$$

The signs adopted require justification :

There is no ambiguity about the result $d\varepsilon = \partial_3$ for the positive direction of the principal normal has been chosen, $\partial_1 = dT$ corresponds to the usual convention in English text-books, whereby,

$$\frac{1}{\sigma\rho^2} = \begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix} = \begin{vmatrix} 1, & 0, & 0 \\ 0, & \frac{1}{\rho}, & 0 \\ -\frac{1}{\rho^2}, & \frac{1}{\rho^2}\frac{d\sigma}{ds}, & \frac{1}{\rho\sigma} \end{vmatrix}$$

vide § 6, where the expansions of the coordinates in terms of the arc are found. Clearly in the standard curve σ is positive when the curve rises above the osculating tangent plane or the positive direction of the tangent, or when the osculating plane, seen in the positive direction of the tangent, turns in the right handed sense. The opposite convention for σ is sometimes adopted by continental writers. Hence for a curve in space

$$\partial_1 = dT,\ \partial_2 = o,\ \partial_3 = d\varepsilon \quad \dots \quad \dots \quad \dots \ (1)$$

The angle through which the principal normal turns is from the second equation of § 3,

$$\sqrt{\partial_1^2 + \partial_3^2} \qquad \text{(Frost, § 652).}$$

§ 5. If the direction cosines of the tangent, principal normal and binormal are (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) referred to axes fixed in space, then the straight line whose direction cosines are (l_1, l_2, l_3) is fixed relatively to the moving axes;

$$\therefore \quad dl_1 - l_2\, d\varepsilon = o.$$
$$dl_2 - l_3\, dT + l_1 d\varepsilon = o.$$
$$dl_3 + l_2\, dT = o.$$
$$\therefore \quad \frac{dl_1}{ds} = \frac{l_2}{\varrho}, \quad \frac{dl_2}{ds} = \frac{l_3}{\sigma} - \frac{l_1}{\varrho}, \quad \frac{dl_3}{ds} = -\frac{l_2}{\sigma}$$

and similarly

$$\frac{dm_1}{ds} = \frac{m_2}{\delta}, \quad \frac{dm_2}{ds} = \frac{m_3}{\sigma} - \frac{m}{\varrho}, \quad \frac{dm_3}{ds} = -\frac{m_2}{\sigma}.$$
$$\frac{dn_1}{ds} = \frac{n_2}{\varrho}, \quad \frac{dn_2}{ds} = \frac{n_3}{\sigma} - \frac{n_1}{\varrho}, \quad \frac{dn_3}{ds} = -\frac{n_2}{\sigma}.$$

These are Frenet's formulae, sometimes attributed to Serret.

§ 6. The expansion of the coordinates of a point on a curve in terms of the arc can be obtained by means of Frenets formulae.

Writing $\frac{1}{\varrho} = a, \frac{1}{\sigma} = k$ we have

$$\frac{dl_1}{ds} = l_2 a, \quad \frac{dl_2}{as} = kl_3 - al_1, \quad \frac{dl_3}{ds} = -kl_2.$$

Hence
$$\frac{d^2 l_1}{ds^2} = l_2 \frac{da}{ds} + akl_3 - a^2 l_1.$$

and
$$\frac{d^3 l_1}{ds^3} = -3a\frac{da}{ds}l_1 + \left(\frac{d^2 a}{ds^2} - ak^2 - a^3\right) l_2 + \left(k\frac{da}{ds} + \frac{d}{ds}(ak)\right) l_3.$$

For the expansion of x we write $l_1 = 1$, $l_2 = l_3 = o$, and obtain

$$x = s - \frac{a^2}{\underline{|3}} s^3 - \frac{s^6}{\underline{|4}} 3a\frac{da}{ds} + \frac{s^5}{\underline{|5}} \left\{ -3\frac{d}{ds}\left(a\frac{da}{ds}\right) - a\frac{d^2 a}{ds^2} + a^2k^2 + a^4 \right\}$$
$$= s - \frac{s^3}{6\varrho^2} + \frac{s^4}{8\varrho^3}\frac{d\varrho}{ds} + \frac{s^5}{\underline{|5}\,\varrho^4}\left\{1 - 11\left(\frac{d\varrho}{ds}\right)^2 + 4\rho\frac{d^2\varrho}{ds^2} + \frac{\varrho^2}{\sigma^2}\right\} + \ldots$$

For the expansions of y and z, we write $l_1 = o$, $l_2 = 1$, $l_3 = o$ and $l_1 = l_2 = o$, $l_3 = 1$.

$$\therefore\; y = \frac{s^3}{2\varrho} - \frac{s^2}{6\varrho^2}\frac{d\varrho}{ds} - \frac{s_4}{\underline{|4}\,\varrho^3}\left(1 - 2\left(\frac{d\varrho}{ds}\right)^2 + \varrho\frac{d^2\varrho}{ds^2} + \frac{\varrho^2}{\sigma^2}\right) +$$
$$\frac{3s^5}{\underline{|5}\,\varrho^4}\left\{2\frac{d\varrho}{ds} - 2\left(\frac{d\varrho}{ds}\right)^3 + 2\varrho\frac{d\varrho}{ds}\frac{d^2\varrho}{ds^2} + \frac{\varrho^2}{\sigma^2}\frac{d\varrho}{ds} + \frac{\varrho^3}{\sigma^3}\frac{d\sigma}{ds}\right\} + .$$

and
$$z = \frac{s^3}{6\varrho\sigma} \cdot \frac{2s^4}{\underline{|4}\,\varrho^2\sigma^2}\left(\sigma\frac{d\varrho}{ds} + \varrho\frac{d\sigma}{ds}\right) + \frac{s^5}{\underline{|5}\,\varrho^3\sigma^3}\left\{6\sigma^2\left(\frac{d\varrho}{ds}\right)^2 - 3\varrho\sigma^2\frac{d^2\rho}{d\sigma^2}\right.$$
$$\left. -\rho^2 - \sigma^2 + 3\rho\sigma\frac{d\varrho}{ds}\frac{d\sigma}{ds} + 2\varrho^2\left(\frac{d\sigma}{ds}\right)^2 - \rho^2\sigma\frac{d^2\sigma}{ds^2}\right\} + .$$

§ 7. *To obtain the equation of the cone of revolution of closest contact with its vertex at the origin.*

By substituting the values of x, y, z in § 6 in the equation $ax^2+by^2+cz^2+2hxy+2gzx+2fyz=0$, and equating to zero the coefficients of s^2, s^3, s^4, we find,

$$a=0,\ h=0,\ \frac{b}{4\varrho^2}+2g\frac{1}{6\varrho\sigma}=0,$$

and for a cone of revolution, since $h=0$, either $g=0$ or $f=0$; and since $g=0$ would reduce the cone to two planes, we have $f=0$ and $b(b-c)=g^2$. Thus the cone is

$$4\varrho y^2+\left(4\varrho-\frac{9\sigma^2}{4\delta}\right)z^2-6\sigma xz=0.$$

If α is the semivertical angle and l, m, n the axis of this cone we identify it with $(x^2+y^2+z^2)\cos^2\alpha=(lx+my+nz)^2$, whence $l=\cos\alpha$, $m=0$, and the equation $(y^2+z^2)\cos^2\alpha=n^2z^2+2n\,xz\cos\alpha$ gives by comparison $\tan\alpha=\frac{3\sigma}{4\varrho}$ and the $d.\,c$ of the axis are proportional to $(4\varrho, 0, 3\sigma)$.

It may be noticed that the cone of revolution touching three consecutive osculating planes is not identical with this.

Similarly, *for the sphere of closest contact*, by substitution in $x^2+y^2+z^2-2ax-2by-2cz=0$, we easily get $a=0$, $b=\varrho$, $c=\sigma\frac{d\rho}{ds}$.

Thus the centre of the osculating sphere is $\left(0, \varrho, \sigma\frac{d\varrho}{ds}\right)$ and the radius of spherical curvature is given by

$$R^2=\varrho^2+\sigma^2\left(\frac{d\rho}{ds}\right)^2.$$

§ 8. I consider one more problem of this nature.

A circular cylinder has contact of the third order with a twisted curve at a point at which the curvature and torsion are $1/\varrho$, $1/\sigma$. *Show that the generator of the cylinder which passes through the point is one of the generators of the cubic cone*

$$\left(\frac{\rho}{\sigma}y+\frac{d\rho}{ds}z\right)(y^2+z^2)-3xyz=0. \qquad \text{(M.T: 1897)}$$

If $(l,m,n,)$ are the $d.c$'s of the axis of the cylinder, its equation is

$$(lx+my+nz)^2-x^2-y^2-z^2+2ux+2vy+2wz=0,$$

where $lu+mv+nw=0$.

For, in this case there is a line of centres $\frac{x-u}{l}=\frac{y-v}{m}=\frac{z-w}{n}$

Thus

$$\left(ls+m\frac{s^2}{2\rho}\right)^2-s^2+2u\left(s-\frac{s^3}{6\rho^2}\right)+2v\left(\frac{s^2}{2\rho}-\frac{s^3}{6\rho^2}\frac{d\rho}{ds}\right)+2w\frac{s^3}{6\rho\sigma}=0$$

and

$$lu+mv+nw=0.$$

$$\therefore \quad u=0,\ l^2=1-\frac{v}{\rho},\ lm=\frac{v}{3\rho}\frac{d\rho}{ds}-\frac{1}{3}\frac{w}{\sigma},\ mv+nw=0.$$

Thus the locus of the generator $\frac{l}{x}=\frac{m}{y}=\frac{n}{z}=\frac{1}{\sqrt{x^2+y^2+z^2}}$ is given by

$$\frac{xy}{x^2+y^2+z^2}=\frac{1}{3\rho}\frac{d\rho}{ds}\frac{\rho(y^2+z^2)}{x^2+y^2+z^2}+\frac{1}{3\sigma}\frac{y}{z}\frac{\rho(y^2+z^2)}{x^2+y^2+z^2},$$

which is a cubic cone as given.

§ 9. A simple application.

At any point P of a cycloidal arc whose vertex is A, PQ is drawn perpendicular to the plane of the arc so that PQ=k times the length of the art AP. If ρ_0 is the radius of curvature of the cycloid at P, prove that the radius of torsion at Q of the locus of Q is $\left(k+\frac{1}{k}\right)\rho_0$ (Oxford 1908).

If ϕ is the angle the tangent at P makes with the tangent at the vertex A, the coordinates of P are $a(2\phi+\sin 2\phi)$, $a(1-\cos 2\phi)$ whence $AP=4a\sin\phi$, and $\rho_0=4a\cos\phi$.

Take moving axes at P *viz.*, the tangent at P, the normal at P and the perpendicular to the plane of the cycloid.

Then $\partial_1=0$, $\partial_2=0$, $\partial_3=d\phi$, $\xi=4a\cos\phi\, d\phi$, $\eta=0$, $\zeta=0$.

The coordinates of Q are $(0, 0, 4ak\sin\phi)$

Whence by §2 (1).

$$\Delta s^2=(4a\cos\phi\, d\phi)^2+(4ak\cos\phi\, d\phi)^2.$$

The direction cosines of the tangent are $\frac{1}{\sqrt{1+k^2}}, 0, \frac{k}{\sqrt{1+k^2}}$

and by § 2 (2)

$$\Delta\varepsilon^2=0+\left(\frac{d\phi}{\sqrt{1+k^2}}\right)^2+0.$$

The radius of curvature of Q is therefore $4a(1+k^2)\cos\phi$.

For the binormal $\lambda+k\nu=0$, $\mu=0$.

Thus the *d.c's* of the binormal are $\left(\frac{k}{\sqrt{1+k^2}}, 0, -\frac{1}{\sqrt{1+k^2}}\right)$

and $\Delta T^2=\frac{k^2}{1+k^2}d\phi^2$ by § 3 (4).

whence $\sigma=4a\cos\phi\left(k+\frac{1}{k}\right)$

The curve is of course a geodesic on the cycloidal cylinder.

§ 10. *On the locus of the centre of spherical curvature.*

I shall not assume the results established in § 7 for the position of the centre of spherical curvature.

Taking the tangent, normal and binormal as axes, we have

$$\xi = ds,\ \eta = 0,\ \zeta = 0,\ \partial_1 = dT_1,\ \partial_2 = 0,\ \partial_3 = d\varepsilon.$$

The corresponding elements of arc, contingent angle, and torsion angle will be denoted by ds_1, $d\varepsilon_1$, $d\sigma_1$ and the curvature and torsion by $\frac{1}{\rho_1}$, $\frac{1}{\sigma_1}$

The centre of spherical curvature is the intersection of three consecutive normal planes.

The normal plane is $x=0$, the consecutive plane by § 3 (6) is

$$x - ds + y\ d\varepsilon = 0.$$

Thus the line of intersection is $x=0$, $y=\rho$, the so called polar line.

The consecutive plane to $y-\rho=0$ is

$$y + z\ dT - x\ d\varepsilon - \rho - d\rho = 0.$$

Whence for the centre of spherical curvature $x=0$, $y=\varrho$, $z=\sigma\frac{d\varrho}{ds}$

Hence the centre is $\left(0,\ \varrho,\ \sigma\frac{d\varrho}{ds}\right)$ and $R^2 = \varrho^2 + \sigma^2\left(\frac{d\varrho}{ds}\right)^2$

For the radii of curvature and torsion of this locus we have

$$dp_1^2 = (ds - \varrho d\varepsilon)^2 + \left(d\varrho - \sigma\frac{d\varrho}{ds}\ dT\right)^2 + \left(\frac{d}{ds}\left(\sigma\frac{d\varrho}{ds}\right)ds + \varrho\, dT\right)^2$$

$$\therefore \frac{ds_1}{ds} = \frac{\varrho}{\sigma} + \frac{d}{ds}\left(\sigma\frac{d\varrho}{ds}\right)$$

The tangent is (0, 0, 1) and the consecutive tangent is $(0, -dT, 1)$

$\therefore d\varepsilon_1 = dT$ and the radius of curvature $= \frac{ds_1}{d\varepsilon_1} = \varrho + \sigma\frac{d}{ds}\left(\sigma\frac{d\varrho}{ds}\right)$

The binormal is perpendicular to the two tangents and is therefore the straight line (1, 0, 0).

$\therefore$ the consecutive binormal is $(1, d\varepsilon, 0)$

$\therefore dT_1 = d\varepsilon$, and the radius of torsion $= \frac{ds_1}{dT_1} = \varrho\left\{\frac{\varrho}{\sigma} + \frac{d}{ds}\left(\sigma\frac{d\varrho}{ds}\right)\right\}$

Thus considering a curve C and its locus of centres of spherical curvature C′ we have the following well-known results :

(1) The binormal of each curve is parallel to the tangent of the other.

(2) The principal normals are parallel.

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(3) The contingence angle of the one is equal to the torsion angle of the other.

(4) The product of the curvatures of the two curves is equal to the product of their torsions.

These properties however belong to any two curves which are such that the tangent at any point of the one is parallel to the binormal at a corresponding point of the other.

I proceed to find the shortest distance between two consecutive lines joining a point on a curve to its centre of spherical curvature.

The line joining the origin to the point $\left(0,\ \varrho,\ \sigma\frac{d\varrho}{ds}\right)$ is

$$\frac{x}{0}=\frac{y}{\varrho/R}=\frac{z}{\sigma\frac{d\varrho}{ds}/R} \text{ where } R^2=\varrho^2+\sigma^2\left(\frac{d\varrho}{ds}\right)^2$$

or

$$\frac{x}{0}=\frac{y}{\cos\chi}=\frac{z}{\sin\chi}.$$

The consecutive line is

$$\frac{x-ds}{-\cos\chi\, d\varepsilon}=\frac{y}{\cos\chi-\sin\chi d\chi-\sin\chi dT}=\frac{z}{\sin\chi+\cos\chi d\chi+\cos\chi dT}\cdots(1)$$

If $\lambda,\ \mu,\ \nu$ are the $d,\ c.$ of the shortest distance

$$\mu\cos\chi+\nu\sin\chi=0$$

and $-\lambda\cos\chi d\varepsilon-(\mu\sin\chi-\nu\cos\chi)(d\chi+dT)=0.$

The shortest distance lies on the straight line

$$\frac{x}{\left(\frac{d\chi}{ds}+\frac{1}{\sigma}\right)\varrho\sec\chi}=\frac{y-k\cos\chi}{-\sin\chi}=\frac{z-k\sin\chi}{\cos\chi}$$

where k, the disttance from the point on the curve to the point where the shortest distance intersects the radius of spherical curvature, is to be determined by the condition that this straight line meets (1).

This condition is

$$\begin{vmatrix} ds, & -k\cos\chi, & -k\sin\chi \\ \left(\frac{d\chi}{ds}+\frac{1}{\sigma}\right)\rho\sec\chi, & -\sin\chi, & \cos\chi \\ -\cos\chi\, d\varepsilon, & \cos\chi-\sin\chi(d\chi+dT), & \sin\chi+\cos\chi(d\chi+dT) \end{vmatrix}=0$$

$$\therefore \quad -ds+k\cos\chi\, d\varepsilon+\left(\frac{d\chi}{ds}+\frac{1}{\sigma}\right)\rho\sec\chi.\, k\,(d\chi+dT)=$$

$$\therefore k\left[\frac{\rho}{R}+\frac{R}{\varrho}\left\{\rho\frac{d\chi}{ds}+\frac{\varrho}{\sigma}\right\}^2\right]=\varrho$$

Now $$-\sin\chi\frac{d\chi}{ds}=\frac{1}{R}\frac{d\varrho}{ds}-\frac{\varrho}{R^2}\frac{dR}{ds}$$

$\therefore$ $$\rho\frac{d\chi}{ds}+\frac{\varrho}{\sigma}=\frac{\varrho^2}{\sigma R}\frac{dR}{d\varrho}$$

and $$k=\frac{R\sigma^2}{\sigma^2+\varrho^2\left(\frac{dR}{d\varrho}\right)^2}$$

a result given in a Trinity College Examination of 1899.

The magnitude of the shortest distance is the projection of ds on the line (λ, μ, ν) and is $\left\{ds\rho\cdot\frac{dR}{d\rho}\right\}\Big/\sqrt{\left[\sigma^2+\rho^2\left(\frac{dR}{d\rho}\right)^2\right]}$

In a curve in which the difference between the radii of absolute and spherical curvature is constant, the arcs of the loci of the centres of absolute and spherical curvature, measured between corresponding points are equal.

[Frost Exx XLVII (4)].

For the locus of the centre of absolute curvature $(0, \varrho, 0)$ we have

$$d_1^2=ds\varrho^2+\rho^2 dT^2$$

For the locus of centres of spherical curvature we have just found

$$ds_2^2=\left[\varrho dT+\frac{d}{ds}\left(\sigma\frac{d\varrho}{ds}\right)ds\right]^2$$

and if $R-\varrho=$constant, we have

$$\sqrt{\varrho^2+\sigma^2\left(\frac{d\varrho}{ds}\right)^2}-\delta = \text{constant.}$$

$$\therefore\left[\varrho d\varrho+\sigma\frac{d\varrho}{ds}\frac{d}{ds}\left(\sigma\frac{d\varrho}{ds}\right)ds\right]^2=d\varrho^2\left[\varrho^2+\sigma^2\left(\frac{d\varrho}{ds}\right)^2\right]$$

$$\therefore\left[\varrho dT+\frac{d}{ds}\left(\sigma\frac{d\varrho}{ds}ds\right)\right]^2=\varrho^2 dT^2+d\rho^2$$

whence $ds_1=ds_2$ and $s_1=s_2$.

Lastly we may find the centre and radius of spherical curvature of the locus of $\left(0, \varrho, \sigma\frac{d\varrho}{ds}\right)$

It must be, referred to its own tangent normal and binormal as axes, $\left(0, \rho_1, \sigma_1\frac{d\varrho_1}{ds_1}\right)$; and hence, referred to the original axes, it must be $\left(\sigma_1\frac{d\varrho_1}{ds_1}, \varrho \pm\varrho_1, \pm\sigma\frac{d\varrho}{ds}\right)$, where the ambiguities are to be determined.

I proceed as follows:

The normal plane to the locus of centres of spherical curvature is

$$z=\sigma\frac{d\varrho}{ds}=k$$

The consecutive normal plane is, by § 3 (6),

$$z-y\,dT=k+dk$$

whence $y=-\frac{dk}{dT}=-\sigma\frac{dk}{ds}$, gives the intersection of two consecutive normal planes.

The consecutive line to this is given by

$$y+z dT-x d\varepsilon=-\sigma\frac{dk}{ds}-\frac{d}{ds}\left(\sigma\frac{dk}{ds}\right)ds$$

$$\therefore\qquad \frac{x}{\varrho}-\frac{z}{\sigma}=\frac{1}{\varrho}\frac{d}{ds}\left(\sigma\frac{dk}{ds}\right)$$

$\therefore$ the new centre of spherical curvature is

$$\rho\frac{d\rho}{ds}+\varrho\frac{d}{ds}\left[\sigma\frac{d}{ds}\left(\sigma\frac{d\varrho}{ds}\right)\right],\ -\sigma\frac{d}{ds}\left(\sigma\frac{d\varrho}{ds}\right),\ \sigma\frac{d\rho}{ds}$$

Now $$\sigma_1\frac{d\varrho_1}{ds_1}=\frac{d\varrho_1}{d\varepsilon}=\varrho\frac{d}{ds}\left\{\varrho+\sigma\frac{d}{ds}\left(\sigma\frac{d\varrho}{ds}\right)\right\}$$

Thus the centre is

$$\sigma_1\frac{d\varrho_1}{ds_1},\ \varrho-\varrho_1,\ \sigma\frac{d\varrho}{ds}.$$

The fact that the consecutive tangent had for its direction cosines $(0,-dT,1)$ shows at once that the principal normals are in opposite directions, as in fig. 2, and verifies the correctness of the y-coordinate $\varrho-\varrho_1$.

§ 11. *On the centre of circular curvature.*

I proceed first to find the radius of curvature of the locus of the centre of circular curvature. The system of axes is as in § 4.

The centre is $(0, \varrho, 0)$, whence

$$ds_1^2=(ds-\varrho d\varepsilon)^2+d\varrho^2+(\varrho dT)^2$$
$$=d\varrho^2+\varrho^2 dT^2$$

The $d.c$'s of a tangent are $(0, \cos\phi, \sin\phi)$ where $\tan\phi=\varrho\frac{dT}{d\varrho}$, whence we obtain Frost, Ex XLIV. 8.

Thus $d\varepsilon_1^2=(-\cos\phi d\varepsilon)^2+(-\sin\phi d\phi-\sin\phi dT)^2+(\cos\phi d\phi+\cos\phi dT)$,

which gives $\frac{1}{\varrho_1^2}\left\{\varrho^2+\sigma^2\left(\frac{d\varrho}{ds}\right)^2\right\}=\left(\sigma\frac{d\phi}{ds}+1\right)^2+\frac{\sigma^2\cos^2\phi}{\varrho^2}$.(Frost § 648)

In the case of curves whose radius of circular curvature is constant we have $\phi=\pi/2$ *viz.* the tangent to the locus of the centre of curvature is parallel to the binormal and $\rho_1=\rho$; also by an easy calculation $\sigma\sigma_1=\rho^2$. Each curve is the locus of the centres of circular and spherical curvature of the other. These curves however are not curves on a sphere although the radius of spherical curvature is constant. To see this we may investigate the conditions that a curve may lie on a sphere. Clearly the centre of spherical curvature $\left(0,\ \rho,\ \sigma\frac{d\rho}{ds}\right)$ must be a fixed point.

$$\therefore \qquad ds-\rho d\varepsilon=0,\ d\rho-\sigma\frac{d\rho}{ds}dT=0,\ d\left(\sigma\frac{d\rho}{ds}\right)+\rho dT=0$$

The first two equations are identities but the third gives

$$\left(\sigma\frac{d\rho}{ds}\right)^2+\rho^2=\text{constant}.$$

In case however $\rho=$constant, the last equation reduces to $dT=0$, and the curves are plane. Thus the only curves that lie on a sphere for which $\rho=$constant, are circles.

§ 12. Bertrand's Theorem: *To investigate those curves whose principal normals are also the principal normals of a second curve.*

Consider the curve, locus of the point, $(o,\ k,\ o)$: the consecutive point is $ds-kd\varepsilon,\ k+dk,\ kdT$ and the *d.c.* of the tangent are $ds-kd\varepsilon,\ dk$ kdT which must be perpendicular to $(o,\ 1,\ o)$

$\therefore\ k=$constant

$\therefore$ the *d.c.* of the tangent are $\lambda\left(1-\frac{k}{\rho}\right),\ o,\ \frac{\lambda k}{\sigma}$

where $$\frac{1}{\lambda^2}=\left(1-\frac{k}{\rho}\right)^2+\frac{k^2}{\sigma^2}$$

or $(\sin\chi,\ 0,\ \cos\chi.)$

The consecutive tangent is

$(\sin\chi+\cos\chi\, d\chi,-\cos\chi\, dT+\sin\chi\, d\varepsilon,\ \cos\chi-\sin\chi\, d\chi)$

$\therefore$ for the binormal, which is to be perpendicular to $(0,\ 1,\ 0)$ if this is the principal normal, we have $(\lambda,\ 0,\ \nu)$ as direction cosines.

$$\therefore \qquad \lambda\sin\chi+\nu\cos\chi=0$$
$$\lambda\cos\chi\, d\chi-\nu\sin\chi\, d\chi=0$$

$\therefore$ either $\sin^2\chi=\cos^2\chi$, or $d\chi=0$

Thus χ is constant.

$\therefore$ the curves are given by $1-\frac{k}{\rho}=\frac{k\tan\alpha}{\sigma}$ where α is written for the previous χ.

We may determine the radii of curvature and torsion of this second curve as follows :

$$\frac{1}{\varrho_1^2}\left\{\left(1-\frac{k}{\varrho}\right)^2+\frac{k}{\sigma^2}\right\}=\left(\frac{\cos\alpha}{\sigma}-\frac{\sin\alpha}{\varrho}\right)^2$$

$$\therefore\frac{1}{\varrho_1}=\left(\frac{\cos\alpha}{\sigma}-\frac{\sin\alpha}{\varrho}\right)\frac{\sigma}{k}\cos\alpha.$$

The binormal is $(-\cos\alpha, 0, \sin\alpha)$

$$\therefore\qquad dT_1^2=(-\sin\alpha\, dT-\cos\alpha\, d\varepsilon)^2$$

$$\therefore\qquad \frac{1}{\sigma_1}=\left(\frac{\sin\alpha}{\sigma}+\frac{\cos\alpha}{\varrho}\right)\frac{\sigma}{k}\cos\alpha$$

whence

$$\frac{k}{\varrho_1}+\frac{k\alpha\tan}{\sigma_1}=1.$$

Conversely, if a curve is given by $A/\rho+B/\sigma=1$ it has the above property, that the curve obtained by taking a length A along the principal normal has the same principal normal and the osculating planes are inclined at a constant angle, $\tan^{-1} A/B$, to each other. If however ϱ and σ are both constant, there exist an infinite number of such curves obtained by taking any constant length along the normal. It is well-known that curves for which ϱ=constant, σ=constant are helices on a circular cylinder. If ϱ=constant=A, then B=O and we have the same result as found in § 11.

For these curves

$$\frac{1}{\sigma\sigma_1}=\frac{\alpha\cos}{k}\left(\frac{\sin\alpha}{\sigma}+\frac{\cos\alpha}{\varrho}\right)=\left(\frac{\cos\alpha}{k}\right)^2$$

which is constant. These results are all given in Darboux, Vol I.

§ 13. *Investigate the conditions that the principal normals to a curve may be the binormals to a second curve (Manheim's Theorem).*

The tangent at (o, k, o) has *d. c.*

$$\lambda\left(1-\frac{k}{\varrho}\right),\ \lambda\frac{dk}{ds},\ \lambda\frac{k}{\sigma}$$

and if this is perpendicular to (0, 1, 0) we again have k=constant.

Writing these $(\sin\chi, o, \cos\chi)$ the consecutive tangent is the same as in § 12 and is perpendicular to (0, 1, 0), if

$$-\cos\chi\, dT+\sin\chi\, d\varepsilon=0$$

$$\therefore\ \frac{k}{\sigma^2}=\frac{1}{\varrho}\left(1-\frac{k}{\varrho}\right)$$

which is the necessary condition.

We have

$$\frac{ds_1^2}{ds^2}=\left(1-\frac{k}{\rho}\right)^2+\frac{k^2}{\sigma^2}=\frac{k^2}{\sigma^2}\left(1+\frac{\rho^2}{\sigma^2}\right)$$

$$dT_1^2=d\varepsilon^2+dT^2.$$

$$\therefore \frac{1}{\sigma_1^2}=\frac{\sigma^2}{k^2\rho^2}, \text{ or } \sigma\sigma_1=k\rho.$$

and it may be proved that $\frac{1}{\rho_1}=\frac{k^2}{\sigma^2}\frac{d}{ds}\left(\frac{\rho}{\sigma}\right)$

§ 14. *If the osculating plane to a curve is always tangent to a fixed sphere, centre* O, *then the centre of this sphere lies in the rectifying plane and the ratio of the radius of curvature to the radius of torsion is a linear function of the arc.* (Goursat, Cours d'Analyse, Tome 1 : p. 582).

Take the same axes as is § 4 and let (f, g, h) be the centre of the osphere. Since this is a fixed point.

$$df+ds-gd\varepsilon=o.$$
$$dg-hdT+fd\varepsilon=o.$$
$$dh+gdT=o.$$

The consecutive osculating plane is $z-ydT=o$ and the perpendicular on this from (f, g, h) must be equal to h ;

$\therefore$ $g=o$, which proves the first relation.

$\because$ $h=$constant, and $f+s$ is constant, and

$$\frac{h}{f}=\frac{\sigma}{\rho}=\frac{b}{a-s}$$

which proves the second relation.

It is easily verified that a curve for which $\frac{\rho}{\sigma}=as+b$ is a geodesic on a cone. For, consider the Frenet equations, § 5

$$\frac{dl_1}{ds}=\frac{l^2}{\rho}, \frac{dl_3}{ds}=-\frac{l_2}{\sigma}$$

These give

$$\rho x''+\sigma l'=o$$
$$\rho y''+\sigma m'=o$$
$$\rho z''+\sigma n'=o.$$

where (l, m, n) are written for the direction cosines of the binormal. Integrating $(as+b)\,x''+l'=o$ etc., we obtain

$$(as+b)x'-a(x-x_0)+l=o.$$
$$(as+b)y'-a(y-y_0)+m=o.$$
$$(as+b)z'-a(z-z_0)+n=o.$$

$$\therefore \quad \begin{vmatrix} x', & x-x_0, & l \\ y', & y-y_0, & m \\ z', & z-z_0, & n \end{vmatrix}=o.$$

viz. the tangent, the line joining x_0, y_0, z_0 to the point, and the binormal lie in a plane. Considering the cone through x_0, y_0, z_0 containing the curve; the principal normal of the cone lies in the osculating plane of the curve, the curve is therefore a geodesic on this cone. Conversely consider a geodesic on a cone whose vertex is x_0, y_0, z_0. Since this is a fixed point,

$$dx_0 + ds - y_0 d\varepsilon = o$$
$$dy_0 - z_0 dT + x_0 d\varepsilon = o$$
$$dz_0 + y_0 dT = o$$

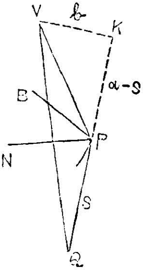
Further, $y_0 = 0$, is the condition the osculating plane contains the normal to the cone,

$\therefore$ $x_0 = a - s$ and $z_0 = b$, where a, b are constants,

and $-z_0\, dT + x_0\, d\varepsilon = o$ gives

$$\frac{\rho}{\sigma} = \frac{a-s}{b}$$

we have therefore a theorem given in Frost L XX, Ex. 1 : *For a geodesic on a cone if an involute is taken* (i) *it lies on a sphere whose centre is the vertex of the cone,* (*ii*) *each tangent to the geodesic makes a constant angle with the corresponding radius of the sphere,* (iii) *the principal normal of the geodesic is perpendicular to the radius of the sphere.*



In the fig. V is the vertex of the cone, P any point on the curve, Q a point on the involute so that PQ $= s$, PN the principal normal, PB the binormal, which lies in the plane VPQ. VK is perpendicular to PQ; then by what has just been proved, VK $= b$, KP $= a - s$, thus VQ$^2 = a^2 + b^2$ $\tan VQP = \frac{b}{d}$ and PN which is perpendicular to the plane VPQ is perpendicular to VQ.

Also, VK is the perpendicular on the osculating plane, thus the osculating plane touches a fixed sphere centre V.

§ 15. *The tangent lines to two twisted curves are conjugate polar lines with regard to a sphere. The radii vectores drawn from the centre of the sphere to corresponding points of the curves are of lengths r, r′ and the perpendiculars on the tangents are p, p′ and the radii of curvature and torsion are ρ, ρ' and σ, σ' prove that*

$$\frac{\rho^2 p^2}{\sigma r^4}=\frac{\rho'^2 p'^2}{\sigma' r'^4} \text{ and } \frac{r^2 r'^2}{\sigma\sigma'}=a^2$$

where a is the radius of the sphere. (M. T. 1898)

Take the tangent, normal and binormal to the first curve as axes, and the sphere as $(x-u)^2+(y-v)^2+(z-w)^2=a^2$.

To determine the conjugate line to $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$ we take the polar plane of any point $(k, 0, 0)$. It is

$$uxk-x(u+k)-vy-wz+u^2+v^2+w^2-a^2=0.$$

Thus the conjugate line is

$$x-u=0.$$

$$ux+vy+wz-u^2-v^2-w^2+a^2=0.$$

The consecutive tangent is

$$\frac{x-ds}{1}=\frac{y}{d\varepsilon}=\frac{z}{0}$$

The polar plane of $(ds+k,\ kd\varepsilon,\ o)$ is

$$x(ds+k)+ykd\varepsilon-u(x+ds+k)-v(y+kd\varepsilon)-wz+u^2+v^2+w^2-a^2=o\ ;$$

thus the consecutive polar line is given by

$$x+yd\varepsilon-u-vd\varepsilon=o$$

$$xds-u(x+ds)-vy-wz+u^2+v^2+w^2-a^2=o.$$

The two polar lines intersect in the point

$$x=u,\ y=v,\ z=\frac{w^2-a^2}{w}.$$

Thus the point and tangent for the second curve are given by

$$\left(u,\ v,\ \frac{w^2-a^2}{w}\right);\ \text{and}\ \frac{x-u}{o}=\frac{y-v}{w}=\frac{z-\frac{w^2-a^2}{w}}{-v}.$$

The constants given in the problem are as follows.

$$r^2=u^2+v^2+w^2,\ r'^2=\frac{a^4}{w^2},$$

$p^2=v^2+w^2$, $p'^2=\frac{a^4}{v^2+w^2}$, this last being the perpendicular from (u, v, w) on the tangent line to the second curve.

Now (u, v, w) is a fixed point.

$$\therefore \quad du+ds-vd\varepsilon=0$$
$$dv-wdT+ud\varepsilon=0$$
$$dw+vdT=0$$

The arc ds' of the second curve is given by

$$ds'^2=(du+ds-vd\varepsilon)^2+(dv-\frac{w^2-a^2}{w}dT+ud\varepsilon)^2+(dw+\frac{a^2}{w^2}dw+vdT)^2$$
$$=\frac{a^4}{w^2}dT^2+\frac{a^4}{w^4}v^2dT^2=\frac{a^4(v^2+w^2)}{w^4}dT^2=\frac{r'^4}{p'^2}dT^2$$

The $d.c$'s of the tangent to the second curve are

$$\left(0, \frac{w}{\sqrt{v^2+w^2}}, -\frac{v}{\sqrt{v^2+w^2}}\right) \text{ or } (0, \cos X, -\sin X)$$

whence

$$d\varepsilon'^2=(-\cos X d\varepsilon)^2+(-\sin X dX+\sin X dT)^2+(-\cos X dX+\cos X dT)^2$$
$$=\cos^2 X d\varepsilon^2+(dX-dT)^2$$

Now $\qquad \tan X=\frac{v}{w}$

$$\therefore \quad \sec^2 X\, dX=\frac{wdv-vdw}{w^2}=\frac{(w^2+v^2)dT-uwd\varepsilon}{w^2}$$

$$\therefore \quad dX=\frac{(w^2+v^2)dT-uwd\varepsilon}{v^2+w^2}$$

$$\therefore d\varepsilon'^2=\frac{w^2d\varepsilon^2}{v^2+w^2}+\frac{u^2w^2d\varepsilon^2}{(v^2+w^2)^2}=\frac{w^2(u^2+v^2+w^2)}{(v^2+w^2)^2}d\varepsilon^2=\frac{p'^2r^2}{r'^2p^2}d\varepsilon^2$$

$$\therefore \quad \frac{1}{\varsigma'^2}=\frac{p'^2.r^2}{r'^2.p^2}\cdot\frac{p'^2}{r'^4}\cdot\frac{\sigma^2}{\varrho^2} \quad \ldots \quad \ldots \quad (1)$$

If (λ, μ, ν) are the direction cosines of the binormal

$$\mu \cos X-\nu \sin X=0$$

and $\quad -\lambda \cos X\, d\varepsilon-\mu \sin X\,(dX-dT)-\nu \cos X\,(dX-dT)=0$

$$\therefore \quad \lambda:\mu:\nu=\frac{dT-dX}{\cos X d\varepsilon}:\sin X:\cos X$$

$$=\frac{uw}{w\sqrt{v^2+w^2}}:\frac{v}{\sqrt{v^2+w^2}}:\frac{w}{\sqrt{v^2+w^2}}$$
$$=u:v:w$$

The binormal is therefore $\left(\frac{u}{r}, \frac{v}{r}, \frac{w}{r}\right)$.

We shall require $dr=d\sqrt{u^2+v^2+w^2}=\dfrac{udu+vdv+wdw}{r}=-\dfrac{uds}{r}$.

By using the relations that express the fact of (u, v, w) being a fixed point, we find

$$drT'^2=ds^2\left(\frac{1}{r}-\frac{u^2}{r^3}\right)^2+\frac{u^2v^2}{r^6}ds^2+\frac{u^2w^2}{r^6}ds^2$$

$$=\frac{v^2+w^2}{r^4}ds^2=\frac{p^2}{r^4}ds^2$$

$$\therefore \qquad \frac{1}{\sigma'^2}=\frac{p^2p'^2}{r^4r'^4}\sigma^2$$

and since $p^2p'^2=a^4$ we have

$$\sigma\sigma'=\frac{r^2r'^2}{a^2}$$

one of the relations required.

Again (1) may be written, by use of $\sigma=\dfrac{r^2r'^2}{\sigma' pp'}$, in the form

$$\frac{\varrho^2}{\varrho'^2}\cdot\frac{\sigma'}{\sigma}=\frac{p'^3}{p^3}\cdot\frac{r^4}{r'^4}$$

which agrees with the first result given. Notice incidentally that the binormal of each curve is parallel to the line joining the centre of the sphere to the corresponding point on the other curve.

(*To be continued.*)

On Fermat's last Theorem.

By T. Hayashi of Tokyo, Japan.

[1.] In the Journal für die reine und angewandte Mathematik Bd. 136 (1909), pp. 293—302, A. Wieferich showed us a necessary condition for the possibility of Fermat's equation.

$$x^n+y^n+z^n=0$$

for an odd prime n, under the supposition that x, y, z are all prime to n. In this note, I will treat the case where one of x, y, z is not prime to n, and find a necessary condition for the possibility of the equation, expressed by the sums of the integral coefficients of the two polynomials Y and Z satisfying the well-known identity

$$4.\frac{\xi^n-1}{\xi-1}=Y^2-(-1)^{\frac{n-1}{2}}n\,Z^2.*$$

* It is already known that there is a connection between this identity and the last theorem of Fermat's. Consult B. Lind, über das letzte Fermatsche Theorem, p. 39, 1910.

[2.] Let the equation

(1.) $$x^n+y^n=z^n$$

be possible for an odd prime n, when z is not prime to n. Let

(2.) $$z=n^p\,\lambda \qquad (p\geqq 1),$$

λ being prime to n.

First of all, we can suppose that x, y, z have no common factor, and therefore x and y are respectively prime to n. Therefore

$$\left.\begin{array}{l} x^n-x\equiv o, \\ y^n-y\equiv o. \end{array}\right\} (\text{mod. } n).$$

and therefore

$$x^n+y^n-(x+y)\equiv O(\text{mod. } n)\cdot$$

But (x^n+y^n), *i.e.* z^n is divisible by n according to our supposition. Hence $x+y$ is divisible by n. Let

(3) $$x+y=n^q\mu \qquad (q\geqq 1),$$

μ being prime to n.

Thus we have

(4) $$\frac{x^n+y^n}{x+y}=\frac{z^n}{n^q\mu}=\frac{\lambda^n}{\mu}n^{np-q}.$$

where λ^n/μ is an integer.

Now

(5) $$x^n+y^n=x^n+(n^q\mu-x)^n$$
$$=n^{nq}\mu^n-n.n^{(n-1)q}\mu^{n-1}x+\binom{n}{2}n^{(n-2)q}\mu^{n-2}x^2$$
$$-+\ldots\ldots\ldots-n.n^q\mu x^{n-1}.$$

But $$x^n+y^n=z^n=n^{np}\lambda^n.$$

Therefore n^{np} is divisible by n^{q+1} and therefore

$$np\geqq q+1.$$

If, however, $np>q+1$, by (5) μx^{n-1} must be divisible by n, which is impossible since μ and x are both prime to n. Therefore

$$np=q+1.$$

Therefore equation (4) becomes

(6) $$\frac{x^n+y^n}{x+y}=n.\frac{\lambda^n}{\mu},$$

Next $$\frac{x^n+y^n}{x+y}=x^{n-1}-x^{n-2}y+x^{n-3}y^2-+\ldots\ldots+y^{n-1}.$$

If we find the quotient of the righthand side divided by $x+y$, then we get

$$\frac{x^n+y^n}{x+y}=\text{a multiple of } (x+y)+nx^{n-1},$$

or by (6)

$$n.\ \lambda^n/\mu=\text{a multiple of } (x+y)+nx^{n-1}.$$

If $n\lambda^n/\mu$ and $x+y$ have a common divisor, this divisor must exactly divide nx^{n-1}. If it exactly divide x, it must exactly divide y, since it is a divisor of $x+y$. But we have supposed that x and y are prime to each other. Therefore the common divisor of $n.\ \lambda^n/\mu$ and $x+y$, *i e.*, $n^q\mu$ is n.

Therefore λ^n/μ and μ are prime to each other. But

$$\mu.\frac{\lambda^n}{\mu}=\lambda^n.$$

Therefore μ and λ^n/μ are both the nth powers of some integers. Let

$$\frac{\lambda^n}{\mu}=k^n.$$

Then by means of (6) we have

$$(7.)\qquad \frac{x^n+y^n}{x+y}=nk^n,$$

i.e., what we can find in B. Lind, über das letzte Fermatsche Theorem, p. 27, 1910.

[**3.**] This equation is quite similar to the equation (11) in my previous note "On the Impossibility of the Indeterminate Equation $x^n+y^n=nz^n$, in which n is an odd prime integer." (*Vide*: J. I. M. S., Vol. III, No. 1.) Hence, proceeding further just as in that note, we arrive at the following theorem:

A necessary condition for the possibility of Fermat's equation

$$x^n+y^n=z^n$$

for an odd prime n and for a value of z divisible by n, is

$$b_0+b_1+b_2+\dots\dots\dots+b_{\frac{n-1}{2}}\equiv 0 \pmod{n^2}$$

where $b_0, b_1, b_2, \dots\dots, b_{\frac{n-1}{2}}$ are the coefficients of the rational integral function Y satisfying the identity

$$4. \quad \frac{\xi^n - 1}{\xi - 1} = Y^2 - (-1)^{\frac{n-1}{2}} n Z^2,$$

in which

$$Y = b_0 \xi^{\frac{n-1}{2}} + b_1 \xi^{\frac{n-1}{2}-1} + b_2 \xi^{\frac{n-1}{2}-2} + \dots\dots\dots + b_{\frac{n-1}{2}},$$

and
$$Z = c_0 \xi^{\frac{n-1}{2}-1} + c_1 \xi^{\frac{n-1}{2}-2} + c_2 \xi^{\frac{n-1}{2}-3} + \dots\dots\dots + c_{\frac{n-1}{2}},$$

when the two expressions

$$\eta = b_0 y^{\frac{n-1}{2}} - b_1 x y^{\frac{n-1}{2}-1} + b_2 x^2 y^{\frac{n-1}{2}-2} - + \dots\dots + (-1)^{\frac{n-1}{2}} b_{\frac{n-1}{2}} x^{\frac{n-1}{2}}$$

and

$$x\zeta = c_0 x y^{\frac{n-1}{2}-1} - c_1 x^2 y^{\frac{n-1}{2}-2} + \dots\dots\dots + (-1)^{\frac{n-1}{2}-1} c_{\frac{n-1}{2}-2}\, x^{\frac{n-1}{2}-1}$$

have such a property that the quadratic form

$$\eta^2 - (-1)^{\frac{n-1}{2}} n (x\zeta)^2$$

admits, as its divisors, only 2 and the numbers of the form $u^2 - (-1)^{\frac{n-1}{2}} n v^2$.

For example, if $n=5$, or $n=13$, the equation is impossible for a value of z divisible by n, as is well-known.

SHORT NOTES.

Note on Indeterminate Equations.

1. Let $a_1x_1+a_2x_2=n$ be an indeterminate equation of the first degree in two unknowns x_1, x_2. To find the *number* of solutions of this equation in integers, *zero* values being included, we may proceed a follows :—

Denote by $I_1(n)$ the number of integral solutions of the equation $a_1x_1=n$. Then $I_1(n-a_2)$ will denote the number of solutions of the equation $a_1x_1+a_2=n$, corresponding to the value 1 of x_2; $I_1(n-2a_2)$ will denote the number corresponding to the value 2 of x_2; and so on. Hence, the total number of solutions of the proposed equation may be written

$$I_1(n)+I_1(n-a_2)+I_1(n-2a_2)+\ldots\ldots\equiv I_2(n)$$

the last term of the series being $I_1(n-p_2a_2)$, where p_2 is the greatest integer in (n/a_2).

2. Similarly, we may write the number of solutions in positive integers of the equation $a_1x_1+a_2x_2+a_3x_3=n$, as

$$I_2(n)+I_2(n-a_3)+I_2(n-2a_3)+\ldots\ldots I_2(n-p_3a_3)\equiv I_3(n)$$

where p_3 denotes the greatest integer in (n/a_3)

3. Generally, the number of positive integral solutions of the equation $a_1x_1+a_2x_2+\ldots\ldots a_rx_r=n$, is written

$$I_{r-1}(n)+I_{r-1}(n-a_r)+I_{r-1}(n-2a_r)+\ldots\ldots I_{r-1}(n-p_ra_r)\equiv I_r(n)$$

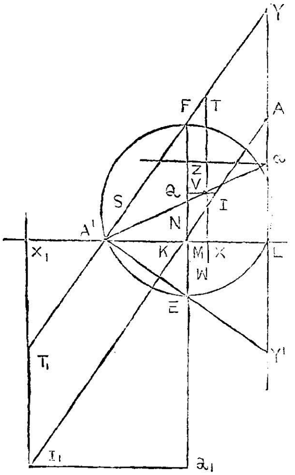
where p_r is the greatest integer in (n/a_r).

JAMESHEDJI EDULJI.

A proof of Feuerbach's Theorem.

1. The line AK bisecting the angle A of the triangle ABC contains I the incentre and I_1 the corresponding excentre. Let A′ be the mid-point of the base, AL the perpendicular on it, and α the mid-point of the segment from A to the orthocentre: then N the mid-point of Aα is the *n.p.* centre. Draw FE the diameter of this circle bisecting A′L, and IQ, I_1Q_1 perpendiculars upon it; and let IX, I_1X_1 meet A_1 F in T, T_1. As BI_1, BI bisect adjacent angles, $I_1K : KI = I_1A : AI$; hence $X_1K : KX = X_1L : LX$. Also $BX_1 = CX$, so that A′ bisects XX_1, and $A'X^2 = A'X_1^2 = A'K.\ A'L$. From these equalities we deduce $A'K : KX = A'X : XL$ and $A'K : KX_1 = A'X_1 : X_1L$. Again the angle KAL is seen

to be $\frac{1}{2}$ A′αL = A′FE, so that A′T is parallel to IK. Therefore we have TI : IX = A′K : KX = A′X : XL ; from which we infer that I lies on a parabola touching A′F, FL at A′, L respectively. Its focus is at N, as NA′ = NF = NL ; its vertex V bisects MF ; and taking VZ = NV, we see that Z is the foot of the axis on the directrix. Thus, finally, IN = QZ = ZM − MQ = NF − IX, and it follows that the incircle touches the *n.p.* circle AαL.



Similarly, $T_1I_1 : I_1X_1 = A'K : KX_1 = A'X_1 : X_1L$, so that I_1 lies on the same parabola. Therefore $I_1N = Q_1Z = ZM + MQ_1 = NF + I_1X_1$, and we see that the excircle touches A′αL externally.

2. The proof may be so arranged as to avoid reference to the parabola. For, ME = NE − NM = NF − NM = MF − 2NM = 2(MV − NM) = 2VN = ZQ + QN, as V is the midpoint of ZN ; and 2 VQ = ZQ − QN.

Now $\frac{A'X}{A'K} = \frac{A'L}{A'X} = \frac{XL}{KX}$, so that A′X.XL = 2A′M.KX ; also the triangles IKX, A′ME are similar, so that IX.ME = A′M.KX.

Hence

$$IQ^2 = MX^2 = A'M^2 - A'X.XL = ME.MF - 2ME.IX$$
$$= ME\,(2MV - 2MQ) = 2VN.2VQ = (ZQ + QN)(ZQ - QN) = ZQ^2 - QN^2.$$

Thus $IQ^2+QN^2=ZQ^2$; that is to say, $IN=ZQ$; and as before we see that the incircle touches the *n.p.* circle,

A similar process applies to the excircle.

3. The triangle $A'\alpha L$ being fixed, A moves on αL, and I, I_1 in the figure lie on the parabola of §1. In this case it is also seen that I_2, I_3 move on another parabola whose focus is N and vertex W (the mid-point of ME), and which touches EA′, EL at A′,L. As A varies continuously the tact-centres will move continuously on the parabolas whose verties are V and W. When A is at α the triangle is right-angled; I, I_1 lie on V at the points where it is crossed by the bisector of A′AL; I_2, I_3 lie on W where it is crossed by the external bisector. As A moves away along $L\alpha$ produced, I, I_1 move towards A′, until when A is at Y, the intersection of AF and $L\alpha$, both coincide with A′. The triangle reduces to Y′A, and I_2, I_3 lie on the second parabola at its intersection with the line through Y parallel to A′E. When A is beyond Y the triangle is imaginary. As A moves from α towards L, I tends towards L, When A is at L, the triangle reduces to BL: I and I_2 coincide with L; I_1, I_3 lie on lines through L parallel to AF, AE, at their intersection with V, W, When A is on αL produced below L, I, I_1 lie on the second parabola, where the line through A parallel to A′E meets it; I_2, I_3 on the first, where the line through A parallel to A′F meets it. As A moves away, I, I_1 tend towards A. until when A is at Y′, both coincide with A. The triangle reduces to Y′A, and I_2, I_3 lie on the first parabola at its intersection with the line through Y′ parallel to A′F.

K. J. Sanjana M.A.

On the face of the Sky for June & July.

(N.B.—Standard time is always referred to).

Sirdereal time at 8 P.M.

	June. H.	M.	S.	July. H.	M.	S.
1	12	35	49	14	34	6
8	13	3	25	15	1	42
15	13	31	1	15	29	18
22	13	58	37	15	56	54
29	14	26	13	16	24	30

From this table the constellations visible during the evenings of June and July can be ascertained by a reference to their position as given in the starcharts.

Phases of the Moon.

	June.		H.	M.		July.		H.	M.	
First Quarter	4	...	3	34	A.M.	... 3	...	2	50	P.M.
Full Moon	12	...	3	21	„	... 11	...	6	23	„
Last Quarter	20	...	2	21	„	... 19	...	11	1	A.M.
New Moon	26	...	6	50	P.M.	... 26	...	1	42	„

The sun

enters Cancer on June 22 at 7-5 P.M. (Summer Solstice)

Planets.

Mercury is a morning star in June attaining its greatest elongation (24°30′ W) on June 1 and superior conjunction on July 4, after which date it is an evening star. It is in conjunction with the Moon on June 26 and on July 27, with Neptime on July 8 and with α Leonis on July 29.

Venus is an evening star throught these months attaining its greatest elongation (45°29′ E) on July 27 at 9-30 P.M.

Mars is in conjunction with the Moon on June 21 at 6-14 A.M. and on July 20 at 1-1 A.M.

Jupiter is very favourably situated for study during these months being in Libra. It is in conjunction with the Moon on June 8 and on July 5. The great red spot will be central on Jupiter's disc on

			H.	M.					H.	M.	
June	11	at	1	3	A.M	July	14	at	3	50	A.M.
„	13	„	3	11	„	„	16	„	5	29	„
„	15	„	4	49	„						
„	18	„	6	28	„						

Satellite III will cross the disc on the following dates.

			H.	M.			H.	M.	
June	1	ingress	5	37	A.M.				
July	6	„	3	33	„	egress	5	10	A.M.
„	13	„	2	26	„	„	4	15	„

Saturn is a morning star near the sun. It is in conjunction with the moon on June 23 and on July 21 at 4-41 A.M.

Uranus continues to retrograde, reaching opposition on July 21. It is in conjunction with the moon on June 15 and on July 12.

Neptune reaches conjunction on July 15. It is in conjunction with the moon on June 27 and on July 25.

Meteor Showers.

The following are the important showers during these months.

Date.		R.A.	D.	Name.
June 1—4	...	333	+27	η Pegasids
June	...	312	+61	α Cepheids
June 10—28	...	335	+57	δ Cepheids
June	...	302	+24	η Sagittids
July 6—31	...	284	−13	λ Aquarids
July	...	269	+48	γ Draconids
July 6—31	...	315	+48	α Cygnids
„ 15—28	...	304	−10	α Capricornids
„	...	280	+57	b Draconids
„	...	30	+36	β Triangulids
„ 23—31	...	47	+43	α—β Perseids
„ 25—30	...	339	−11	δ Aquarids
„	...	339	−27	α Piscis Anshatids

The Perseids continue visible more than a month, during which period their radiant point exhibits a diurnal motion of about 1° to the E. N. E. The places of this point in July are

		R.A,	D.
July 19	...	18°.9	+50°.5
„ 21	...	20°.8	+51°.1
„ 23	...	22°.8	+51°.8
„ 25	...	24°.9	+52°.5
„ 27	...	27°.1	+53°.2
„ 29	...	29°.3	+53°.8
„ 31	...	31°.6	+54°.4

V. Ramesam.

SOLUTIONS.

Questions 179, 192, & 253.

179. (V. RAMASWAMI AIYAR, M.A.) :—Let O be the circum centre P the orthocentre and I the centre of a circle touching the sides of a triangle ABC. If the reflexions of AO, AP in IO, IP, respectively cut at a, show that a lies in the side BC.

If AL, AM be the perpendiculars from A drawn to IO, IP, and X be the middle point of Aa show that XL=XM. Also, show that the circle whose centre is X and radius XL or XM touches the nine points circle of ABC at the corresponding Feuerbach's point.

192. (V. RAMASWAMI AIYAR, M.A.) :—ABC and DEF are two triangles. If the pedal circles of the points D, E, F with respect to the triangle ABC are coaxal then, (1) the pedal circles of the points A, B, C with respect to the triangle DEF are also coaxal and of the same coaxal system ; (2) the triangles ABC and DEF are inscribable in a conic X, and circumscribable to a conic Y, and the director circle of the conic Y is a member of the same coaxal system ; (3) if A′B′C′ and D′E′F′ be *any* two triangles inscribed in X and circumcribed to Y, they are related to one another in the same manner as the triangle ABC and DEF ; (4) if X is a circle, Y is a parabola, and the case becomes that of the elegent theorem of Mr. S. Narayanan, B.A., L.T. (Q. 178).

253. (M. BHIMASENA RAO) :—If any straight line or any conic circumscribing the triangle of reference cuts the cubic whose trilinear equation is $p\alpha\ (\beta^2+\gamma^2)+q\beta(\gamma^2+\alpha^2)+r\gamma(\alpha^2+\beta^2)+s\alpha\beta\gamma=0$ in points D,E,F show that (1) D, E, F and their isogonal conjugates, with respect to the triangle lie three by three on four straight lines ; (2) any conic touching these four lines has a confocal one inscribable in the triangle ; (3) the mutual orthoptic circle of the two conics belongs to a coaxal system and Deduce Mr. V. Ramaswami Aiyar's results (1) and (2) of Q. 192 and Q 179.

Solutions by S. Purushotham and M. Bhimasena Rao.

Question 253.

(1) Let $$l\alpha+m\beta+n\gamma=0$$

be the equation of any straight line cutting the cubic in D $(\alpha_1\,\beta_1\,\gamma_1)$ E $(\alpha_2\beta_2\gamma_2)$, F $(\alpha_3\beta_3\gamma_3)$; then if D,E,F are related as in the question

$$\alpha_3:\beta_3:\gamma_3=\frac{\beta_1\beta_2-\gamma_1\gamma_2}{\beta_1\gamma_2-\beta_2\gamma_1}:\frac{\gamma_1\gamma_2-\alpha_1\alpha_2}{\gamma_1\alpha_2-\gamma_2\alpha_1}:\frac{\alpha_1\alpha_2-\beta_1\beta_2}{\alpha_1\beta_2-\beta_1\alpha_2}$$

since F is the intersection of DE and D′E′ where D′ and E′ are isogonal conjugate of D and E.

Putting
$$\alpha = -\frac{m\beta + n\gamma}{l}$$

we get

$$-p\frac{(m\beta+n\gamma)}{l}(\beta^2+\gamma^2)+q\beta\left\{\gamma^2+\frac{(m\beta+n\gamma)^2}{l^2}\right\}$$
$$+r\gamma\left\{\frac{(m\beta+n\gamma)^2}{l^2}+\beta^2\right\}=0.$$

Hence
$$\frac{\beta_1\beta_2\beta_3}{\gamma_1\gamma_2\gamma_3} = -\frac{n(rn-lp)}{m(qm-lp)}$$

If F is the same as the F of the question, we have after some redection

$$\sum p\left(\frac{\beta_1}{\gamma_1}-\frac{\beta_2}{\gamma_2}\right)\left(-\frac{\gamma_1\gamma_2}{\beta_1\beta_2}\right)=0.$$

i.e.
$$\sum p\left(\frac{\beta_1}{\gamma_1}+\frac{\gamma_1}{\beta_1}\right)-\sum p\left(\frac{\beta_2}{\gamma_2}+\frac{\gamma_2}{\beta_2}\right)=0,$$

and this condition is satisfied, since

$$p\left(\frac{\beta_1}{\gamma_1}+\frac{\gamma_1}{\beta_1}\right)+q\left(\frac{\gamma_1}{\alpha_1}+\frac{\alpha_1}{\gamma_1}\right)+r\left(\frac{\alpha_1}{\beta_1}+\frac{\beta_1}{\alpha_1}\right)+s=0.$$

and
$$p\left(\frac{\beta_2}{\gamma_2}+\frac{\gamma_2}{\beta_2}\right)+q\left(\frac{\gamma_2}{\alpha_2}+\frac{\alpha_2}{\gamma_2}\right)+r\left(\frac{\alpha_2}{\beta_2}+\frac{\beta_2}{\alpha_2}\right)+s=0.$$

Hence F is the intersection of DE and D′E′.

If F′ is the intersection of DE′ and D′E, F′ is the other focus of a conic inscribed in DE D E′ with F as one focus and is therefore the isogonal conjugate of F.

(2) This is the same as Mr. Ramaswamy Iyer's result proved on page 34 of the Journal (February 1910).

(3) Let X be a common point of the director circles of the first system and X′, that of the second. If r and r' be the radii of the director circles of a conic and its confocal, the radius ρ of the mutual orth-optic circle is given by $2\rho^2=\gamma^2+\gamma'^2$. Let O be the centre of the two conics. Then $OX=r$, $OX'=r'$ and $OX^2+OX'^2=2\ (OM^2+MX^2)$ where M is the mid point of XX′.

$$\therefore\ 2\rho^2=\gamma^2+\gamma'^2=OX^2+OX'^2=2\ (OM^2+MX^2)$$
$$\therefore\qquad \rho=OM^2+MX^2$$

Therefore the circle whose centre is M and radius $MX\sqrt{-1}$ cuts orthogonally the mutual orthoptic circle. In other words, the mutual orthoptic circle bisects the circumference of the circle on XX′ as diameter. Since its centre lies on a fixed line, it belongs to a coaxal system.

Question 192.

(1) The mutual orthoptic circle of conics as above one of whose foci is D, is the pedal of D with respect ABC. Similarly if one of the common foci is A, the pedal of A with respect to DEF is the corresponding orthoptic circle. (Since the opposite sides DE, D′E′ subtend equal or supplementary angles at A, A is a possible position for a focus of an inconic of DED′E′).

Similarly the other pedal circles are orthoptic circles. Hence they are all coaxal.

(2) DEF lies on a circum conic because the isogonal conjugates of DEF are in a line.

Hence by a well-known theorem a conic can be inscribed in DEF and ABC. The mutual orthoptic circle in this case is the director circle and hence this director circle is a member of the coaxal system.

Question 179.

In this case DED′E′ reduces to IOIP Hence the first system of conics must touch IP and IO at P and O.

Evidently A and a are the foci of a conic touching IO and IP. (Such a conic is possible since IP and IO are opposite sides of the quadrilateral IPIO and subtend equal angles at A.)

Since IO and IP are tangents to the conic at O and P, L and M lie on the auxiliary circle of the conic, in other words, the circle whose centre is X and radius XL or XM. This auxiliary circle is the mutual orthoptic circle of the two confocal conics whose foci are A and a and which are inscribed in IOIP and ABC. Hence this circle is coaxal with the pedals of I and O with respect to the triangle ABC. Hence the circle with X as centre and XL as radius touches the nine point circle at the Feuerbach's point.

192(3)

Let D′ be any point on X and let α' be its isogonal conjugate. then α' is in $\alpha\ \beta\ \gamma$. *To prove that* $\alpha\ \beta\ \gamma$ *is tangent ial to* Y.

Let the other tangent from α to Y cut D F in β'; then A. (B C, D α, E β') is in involution, and therefore the double rays of A (B C, Dα) are at rt angles. Hence, β coincides with β'.

Draw the other tangent from α'. Let tangents from D′ to Y cut tangents from α' in β', E′ & γ', F′.

Then A (BC, D′ α', E′ β', γ' F′) is a pencil in in volution and the double rays of A (B C, D′ α') are at right angles. Hence, AE′, Aβ'

make equal angles with A B, A C. Similarly, with respect to th othere vertices.

Hence β', γ' are isogonal conjugates of E', F'. Since β', γ' lie in $\alpha\beta\gamma$, E' F' lie on X and D'E'F' is circumscribed to Y.

Keeping D' E' F' fixed and changing ABC into A'B'C', we get the required result.

(4) Since the mutual orthoptic circle is orthogonal to a fixed circle whose centre is the mid point of XX', it is obvious that when the mutual orthoptic circle is a straight line, this line bisects XX'—See sol : of Q. 253 (3) above.

Question 259.

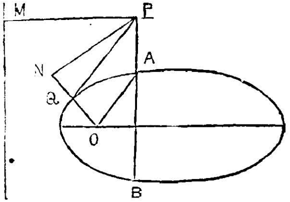
(T. RAJARAM RAO, B.A., B.L.)—AB is a finite straight line and P is a fixed point on AB produced. O is another fixed point on the line which bisects AB at right angles. An ellipse is described passing through A and B with O as one of its foci. If PQ is the tangent to the ellipse, find the locus of Q. What difference will it make in case the curve is a hyperbola ?

Solution (1) by *Zero*, (2) by *S. Narayan B.A., L.T., and N.B. Pendse M.A., LL.B.*

(1) Let the ellipse be described with O for a focus to pass through A, B. Then draw the tangent PQ from P to the ellipse. Join OQ and PN be the perpendicular from P on OQ. Then, if PM be the perpendicular on the directrix. ON = e PM.

$= e$ ($\perp^r$ from A on the directrix.)

$=$ OA.



Hence, the circle (C) with O as centre and radius OA or OB passes through N and touches PN. In other words Q lies in the line joining O to the point of contact of the tangent from P to the circle (C) : that is, the locus of Q is two straight lines.

(2) Reciprocate with respect to the point O. Then the straight line ABP becomes the point K. The points A,B,P become straight lines A,B,P passing through K. The variable conic passing though AB and having O as focus becomes a variable circle touching A, B.

For all the circles of this system K is the centre of similitude. Therefore if the variable circle cuts the line P in two points R and R′, the tangents at R and R′ are parallel to two fixed straight lines and the envelope of these lines are two points at infinity.

Therefore the required locus of Q is a pair of straight lines passing though the fixed point O.

Question 261.

(S. Ramanujan) :—Show that

(a) $$\left(1+\frac{1}{1^3}\right)\left(1+\frac{1}{2^3}\right)\left(1+\frac{1}{3^3}\right)\ldots\ldots ad\ inf. = \frac{1}{\pi}\cosh\frac{\pi\sqrt{3}}{2}$$

(b) $$\left(1-\frac{1}{1^3}\right)\left(1-\frac{1}{2^3}\right)\left(1-\frac{1}{3^3}\right)\ldots\ldots ad\ inf. = \frac{1}{3\pi}\cosh\frac{\pi\sqrt{3}}{2}$$

Solution by N. B. Pendse, M.A., LL.B.

Let $$a = \left(1+\frac{1}{1^3}\right)\left(1+\frac{1}{2^3}\right)\left(1+\frac{1}{3^3}\right)\ldots\ldots$$

$$= \frac{(1^3+1)(2^3+1)(3^3+1)(4^3+1)(5^3+1)\ldots\ldots}{1^3 . 2^3 . 3^3 . 4^3\ldots\ldots}$$

$$b = \frac{(2^3-1)(3^3-1)(4^3-1)(5^3-1)\ldots\ldots}{2^3 . 3^3 . 4^3\ldots\ldots}$$

Then $$\frac{a}{b} = \frac{(1^3+1)(2^3+1)(3^3+1)(4^3+1)\ldots\ldots}{(2^3-1)(3^3-1\ldots\ldots}$$

but $$\frac{(n+1)^3+}{n^3-1} = \frac{(n+2)(n^2+n+1)}{(n-1)(n^2+n+1)} = \frac{n+2}{n-1}$$

$\therefore$ $$\frac{a}{b} = \frac{2\times 9\times(4 . 5 . 6 . 7 . 8\ldots\ldots}{1 . 2 . 3 . 4 . 5 . 6 . 8\ldots\ldots}$$

$$= \frac{2\times 9}{1 . 2 . 3} = 3$$

$\therefore$ $$\frac{a^2}{3} = ab = \left(1+\frac{1}{1^3}\right)\left(1-\frac{1}{2^6}\right)\left(1-\frac{1}{3^6}\right)$$

$$= \left(1+\frac{1}{1^3}\right)\begin{cases}\left(1-\frac{w^2}{2^2}\right)\left(1-\frac{w^2}{3^2}\right)\left(1-\frac{w^2}{4^2}\right)\ldots\ldots \\ \left(1-\frac{w^4}{2^2}\right)\left(1-\frac{w^4}{3^2}\right)\left(1-\frac{w^4}{4^2}\right)\ldots\ldots \\ \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\ldots\end{cases}$$

where w and w^2 are the cube roots of unity.

$$\therefore \quad a^2 = 2\,\frac{\sin w\pi}{w\pi(1-w^2)} \times \frac{\sin w^2\pi}{w^2\pi(1-w^4)} \times \frac{1\,.\,3\,.\,2\,.\,4\,.\,3\,.\,5\,.\,4\,.\,6\,.\,5\,.\,7\,.\,\ldots\ldots}{2^2\,.\,3^2\,.\,4^2\ldots\ldots}$$

$$= \frac{2\sin w\pi \sin w^2\pi}{\pi^2 \times 3} \times \frac{1\,.\,2}{2^2}$$

$$= \frac{\sin^2 w\pi}{3\pi^2},$$

because $w\pi$ and $w^2\pi$ are supplements and their sines are equal.

Hence $$a = \frac{\sin w\pi}{\pi} = \frac{1}{\pi}\sin\left(\frac{\pi}{2} - \frac{i\pi\sqrt{3}}{2}\right)$$

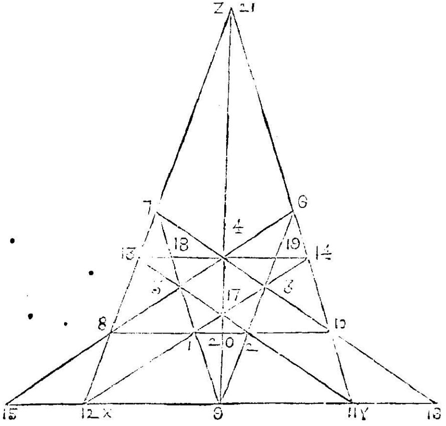
$$= \frac{1}{\pi}\cos\frac{i\pi\sqrt{3}}{2} = \frac{1}{\pi}\cosh\frac{\pi\sqrt{3}}{2},$$

and $$b = \frac{a}{3} = \frac{1}{3\pi}\cosh\frac{\pi\sqrt{3}}{2}.$$

Question 264.

(Selected) —Plant 21 trees in a park so that they should form 12 straight rows with 5 trees in every row.

Solution by Mukund Lal, M.A.



Let 1, 2, 3, 4, 5 be a regular pentagon. Produce its sides to meet in 6, 7, 8, 9 and 10. Through 9 draw a line XY parallel to 8, 1, 2, 10 and let 7, 8 and 10, 6 meeting in Z cut XY in 12 and 11. It can be easily shown that 11, 2, 5 are in one line and so are 12, 1, 3. Let 2, 5 and 1, 3 cut XZ, YZ and each other in 13, 14, and 17 respectively. Then 13, 4, 14 can be shown to be in one line, which cuts 7, 9 and 6, 9 in 18 and 19. Produce 6, 4, 5, 8 and 7, 4, 3, 10 to cut XY in 15 and 16. Again 9, 17, 4, 21 cuts 8, 1, 2, 10 in 20. The rest is obvious.

Question 265.

(ZERO) :—The corners of a regular polygon of $2n$ sides are joined to the centre O, and any transversal cuts the joining lines in ABC......s Prove that $\Sigma(OA^2)=n^2.OM^2$ where M is the mean centre of A,B,C.......

Solution (1) *by Mukhand Lal, M.A., and Solution* (2) *by N. B. Pendse, M.A., LL.B., and T. Rajaram Rao, B.A., B.L.*

[*Note*: The transversal cuts the joining lines in n points only.]

(1) We have

$$\cot\alpha+\cot\left(\alpha+\frac{\pi}{n}\right)+\cot\left(\alpha+\frac{2\pi}{n}\right)++\cot\left(\alpha+\frac{n-1}{n}\pi\right)=n\cot n\alpha \quad (1)$$

(See *Hobson's* Trigonometry, page 118.)

Differentiating

$$\operatorname{cosec}^2\alpha+\operatorname{cosec}^2\left(\alpha+\frac{\pi}{n}\right)+\ +\operatorname{cosec}^2\left(\alpha+\frac{n-1}{n}\pi\right)=n^2\operatorname{cosec}^2 n\alpha \quad (2)$$

Now, if P be the foot of the perpendicular from O on the transversal, then evidently

$$PA+PB+PC+\dots \qquad =n\ PM.$$

$$\therefore\ OP\cot\alpha+OP\cot\left(\alpha+\frac{\pi}{n}\right)+\dots\dots \qquad =n\ OP\cot OMP$$

where α stands for OAP.

Hence, from (1) $\qquad \cot n\alpha=\cot OMP$............(3)

Again, $\qquad OM\sin OMP=OP=OA\sin\alpha$

$$=OB\sin\left(\alpha+\frac{\pi}{n}\right)$$

$$=OC\sin\left(\alpha+\frac{2\pi}{n}\right)$$

$$\therefore\ \Sigma(OA^2)=OM^2\sin^2 OMP\left\{\operatorname{cosec}^2\alpha+\operatorname{cosec}^2\left(\alpha+\frac{\pi}{n}\right)+\operatorname{cosec}^2\left(\alpha+\frac{2\pi}{n}\right)+\dots+\operatorname{cosec}^2\left(\alpha+\frac{n-1\pi}{n}\right)\right\}$$

$$=OM^2\sin^2 OMP\times n^2\operatorname{cosec}^2 n\alpha \text{ from (2)}$$

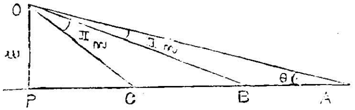
$$=n^2\ OM^2 \qquad \text{from (3)},$$

(2) **From O draw OP perpendicular to ABC. Let OP =** x

$$\Sigma OA^2 = nx^2 + \Sigma PA^2$$

$$= nx^2 + x^2\cot^2\theta + x^2\cot^2\left(\theta + \frac{\pi}{n}\right)$$

$$+ x^2\cot^2\left(\theta + \frac{2\pi}{n}\right) + \ldots\ldots n \text{ terms}$$



Also $$nPM = x\cot\theta + x\cot\left(\theta + \frac{\pi}{n}\right) \ldots\ldots n \text{ terms}$$

Now, $$\cot\theta + \cot\left(\theta + \frac{\pi}{n}\right) + \ldots\ldots = n\cot n\theta$$

$\therefore$ $$PM = x\cot n\theta$$

$\therefore$ $$\Sigma PA^2 = \{ n^2\cot^2 n\theta + n(n-1) \} n^2$$

and $$\Sigma OA^2 = x^2n^2(\cot^2 n\theta + 1)$$

$$= n^2PM^2 + n^2x^2 = n^2OM^2$$

QUESTIONS FOR SOLUTION.

290. (PROF. K. J. SANJANA) :—Construct a triangle, having given the circumcircle, the orthocentre, and that one side passes through a given point: and examine the number of solutions. [Instead of the orthocentre, either the centroid or the nine-point centre may be given.]

291. (M. BHIMASENA RAO) :—ABC and DEF are two triangles inscribed in a circle. Shew that the triangles formed by the pedal lines of A, B, C with respect to DEF and by the pedal lines of D, E, F with respect to ABC are inscribed in another circle whose centre is equidistant from, and collinear with, the orthocentres of ABC and DEF.

[This is an extension of Q : 178 by Mr. S. Narayanan, B.A., LT.]

292. (MUKAND LAL, M.A.) :—Forces $P(\tan\frac{1}{2}A - \tan\frac{1}{2}B)$ $P(\tan\frac{1}{2}B - \tan\frac{1}{2}C)$, $P(\tan\frac{1}{2}C - \tan\frac{1}{2}A)$ act along the sides AB, BC, CA of a triangle ABC. Prove that their resultant is $2P\sqrt{(1-8\sin\frac{1}{2}A.\sin\frac{1}{2}B.\sin\frac{1}{2}C)}$, and give a geometrical construction for the resultant.

293. (J. C. SWAMINARAYAN, M.A.) :—

Shew that in any triangle ABC

$$c^n = a^n \cos nB + n\,a^{n-1}b\cos\{(n-1)B - A\}$$

$$+\frac{n.n-1}{1.2}a^{n-2}b^2\cos\{(n-2)B-2A\} + \quad . \quad . \quad . \quad . \quad + b^n\cos nA.$$

n being a positive integer.

294. (S. RAMANUJAN) :—Shew that

$$\frac{e^x}{2} = 1 + \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \quad . \quad . \quad . \quad \frac{x^x}{\underline{|x}}\theta$$

where θ lies between $\frac{1}{2}$ and $\frac{1}{3}$.

295. (S. RAMANUJAN) :—If $\alpha\beta = \pi$, shew that

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}dx}{\cosh\alpha x} = \sqrt{\beta}\int_0^\infty \frac{e^{-x^2}dx}{\cosh\beta x}$$

296. (J. C. SWAMINARAYAN, M.A.) :—

If $x^2+y^2+z^2 = x'^2+y'^2+z'^2 = r^2$ find the limiting value of

$$\frac{\lambda(yz'-y'z)+\mu(zx'-z'x)+\gamma(xy'-x'y)}{\sqrt{\{(yz'-y'z)^2+(zx'-z'x)^2+(xy'-x'y)^2\}}}$$

when $x=x'$, $y=y'$, $z=z'$,

297. (D. D. Kapadia, M.A.; B.Sc) :—

Prove that (the determinant being of the nth order)

$$\begin{vmatrix} a & b & b & \cdots & \cdots & b & b & b \\ b & a & b & \cdots & \cdots & b & b & b \\ a & a & b & \cdots & \cdots & a & a & a \\ \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ a & a & a & \cdots & \cdots & b & a & a \\ b & b & b & \cdots & \cdots & b & a & b \\ b & b & b & \cdots & \cdots & b & b & a \end{vmatrix} = (b-a)^{n-4}\{(n-5)a-3b\}$$

where, in the first, second, last but one, and last rows there are $(n-1)$ b's, and only one b in the other rows.

298. (M. T Naraniengar M.A.) :— Find the parabolic asymptote of the curve $(y-x)^2(y+x)=2ax^2$, and discuss the correctness of the solution given on p. 201, Edward's *Diffl. Calc.* (1896).

299. (M. T. Naraniengar M.A.) :—With the usual notation shew that in any triangle ABC, II_1 and I_2I_3 subtend supplementary angles at either of the points of the circum circle diametrically opposite to B or C.

300. (Zero) :—If A, B, C be three points on a circle, find a point M on the circle such that MA, MB, MC are equal to the sides of another triangle inscribed in the same circle.

301. (D. D. Kapadia M.A., B.Sc.) :—Prove that in the neighbourhood of a regular point of a plane curve, the equation to the curve referred to the tangent and normal at the point as axes of x and y may be written in either of the forms

$$y=\frac{x^2}{2\rho}-\frac{\rho'}{6\rho^2}x^3+(2+2\rho'^2-\rho\rho'')\frac{x^4}{24\rho^3}+\ldots\ldots$$

$$x=(2\rho y)^{\frac{1}{2}}+\frac{1}{3}\rho' y+\left(\frac{1}{12}\rho\rho''-\frac{1}{4}-\frac{1}{36}\rho'^2\right)\left(\frac{2}{\rho}\right)^{\frac{1}{2}}y^{\frac{3}{2}}+\ldots\ldots$$

where ρ, ρ', ρ'' denote the radius of curvature and its first and second rates of variation along the arc of the curve, at the given point.

Hence find the locus of the centres of conics having four point contact with the curve at the point and express the condition that the conic of closest contact should be a parabola. (Math. Tripos 1910.)

302. (D. D. KAPADIA M.A., B.Sc.):—On two edges AB, CD of a skew quadrilateral ABCD points P, Q are taken so that AP : PB=DQ : QC. Prove, by taking DA, DB, DC as axes of x, y, z, that PQ is a generator of a hyperbolic paraboloid the equation of which can be put down in the form

$$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{y}{b}+\frac{z}{c}\right)=\frac{y}{b}.$$

Hence show how a model of a hyperbolic paraboloid may be constructed with the aid of cardboard, and thread. How is the second system of generators to be constructed ?

301. (A. C. L. WILKINSON, M.A., F.R.A.S.):—Prove that the following expressions are true : the inverse tangents, in all cases, denoting angles between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

(1) $$\sum_1^\infty \tan^{-1}\frac{4\sqrt{2}(n^2+3)}{(n^2-1)^2}=2\pi-\tan^{-1}\frac{2\sqrt{2}}{3}.$$

(2) $$\sum_1^\infty \tan^{-1}\frac{8\sqrt{2}(n^2-6)}{(n^2+2)^2}=\tan^{-1}12\sqrt{2}+\tan^{-1}\frac{6\sqrt{2}}{3}$$

(3) $$\sum_1^\infty \tan^{-1}\frac{4n^2}{n^4-4}=\frac{\pi}{2}.$$

(4) $$\sum_1^\infty \tan^{-1}\frac{16(n^2-8)}{n^2(n^2+16)}=\frac{\pi}{4}.$$

(5) $$\sum_1^\infty \tan^{-1}\frac{4(n^2-A^2)}{(n^2+A^2)^2-4}=\tan^{-1}\frac{2}{A^2}.$$

302. (S. NARAYANAN, B.A., L.T.) :—If δ be the circle described on the base BC of a triangle ABC as diameter, and σ the excosine circle corresponding to A, prove *geometrically* that (1) the pedal triangle of any point on σ is right angled ; and (2) the isogonal transformation of σ is δ.

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